① $a_n \to 0$; $\frac{1}{n^2+2} < \frac{1}{100} \Rightarrow 100 < n^2+2 \Rightarrow 98 < n^2 \Rightarrow n(\varepsilon) = \{\sqrt{98}\} = 9$

$b_n \to 2$; $\left|\frac{2n+1}{n+3} - 2\right| = \left|\frac{\cancel{2n}+1-\cancel{2n}-6}{n+3}\right| = \frac{5}{n+3} < \frac{1}{100} \Rightarrow 500 < n+3 \Rightarrow 497 < n \Rightarrow n(\varepsilon) = 497$

$c_n \to 3$; $\left|\sqrt{9+\frac{1}{n}} - 3\right| < \frac{1}{100} \Rightarrow \sqrt{9+\frac{1}{n}} < \frac{301}{100} \Rightarrow 9 + \frac{1}{n} < \left(\frac{301}{100}\right)^2 \Rightarrow \frac{1}{301^2-9} < n \Rightarrow n(\varepsilon) = \left\{\frac{1}{0{,}0601}\right\} = 16$

② $a_n \to -\infty$; $\frac{b_{n+1}}{b_n} = 2\cdot\left(\frac{n}{n+1}\right)^{20} \to 2 > 1 \Rightarrow b_n \to \infty$; $c_n = \left[\left(1+\frac{7}{3n-2}\right)^{\frac{3n-2}{7}}\right]^{\frac{28n}{3n-2}} \to e^{28/3}$; $d_n \to 1$

③ $\lim\limits_{x\to\infty} \frac{(\sqrt{x^2+x}-x)(\sqrt{x^2+x}+x)}{\sqrt{x^2+x}+x} = \lim\limits_{x\to\infty} \frac{x}{\sqrt{x^2+x}+x} = \frac{1}{2}$; $\lim\limits_{x\to-\infty} \left(\overbrace{\sqrt{x^2+x}}^{\to+\infty} + \overbrace{(-x)}^{\to+\infty}\right) = +\infty$; $\lim\limits_{x\to 0} \frac{3x}{\sin x} = 3$

$\lim\limits_{x\to 0} \frac{\tan x}{\sin 5x} = \frac{1}{5}$; $\lim\limits_{x\to -3} \frac{x^2-9}{x+3} = \lim\limits_{x\to-3} \frac{(x-3)\cancel{(x+3)}}{\cancel{x+3}} = -6$

④ a.) $\lim\limits_{x\to 1} (x+3) = 4 \neq \lim\limits_{x\to 1} \frac{x^2-1}{x-1} = \lim\limits_{x\to 1}(x+1) = 2 \Rightarrow$ finite jump at $x=1$

b.) $\lim\limits_{x\to 0} 3x^2 = 0 = \lim\limits_{x\to 0} \frac{\sin^2 x}{x} = \lim\limits_{x\to 0} \underbrace{(\sin x)}_{\to 0}\underbrace{\left(\frac{\sin x}{x}\right)}_{\to 1} \Rightarrow$ no points of discontinuity

c.) $\lim\limits_{x\to 3^-} \frac{x}{x-3} = \frac{3}{0^-} = -\infty$ infinite jump at $x=3$

⑤ a1.) $\lim\limits_{h\to 0} \frac{(x+h)^2 - 2(x+h) - x^2 + 2x}{h} = \lim\limits_{h\to 0} \frac{2xh + h^2 - 2h}{h} = 2x-2 = f'(x)$

a2.) $\lim\limits_{h\to 0} \frac{\sqrt{x+h+3} - \sqrt{x+3}}{h} = \lim\limits_{h\to 0} \frac{\cancel{x}+h+\cancel{3}-\cancel{x}-\cancel{3}}{h(\sqrt{x+h+3}+\sqrt{x+3})} = \frac{1}{2\sqrt{x+3}} = f'(x)$

a3.) $\lim\limits_{h\to 0} \frac{\frac{2}{x+h} - \frac{2}{x}}{h} = \lim\limits_{h\to 0} \frac{\cancel{2x} - \cancel{2x} - 2h}{h\cdot x(x+h)} = \frac{-2}{x^2} = f'(x)$

b1.) $(x^2\cdot\sin 2x)' = 2x\cdot\sin 2x + x^2\cdot\cos 2x\cdot 2$; b2.) $\left(\frac{\cos x}{x+\ln x}\right)' = \frac{-\sin x(x+\ln x) - \left(1+\frac{1}{x}\right)\cos x}{(x+\ln x)^2}$

b3.) $\left(\sqrt{x^3+5x+2^x}\right)' = \frac{1}{2}\cdot\left(x^3+5x+2^x\right)^{-1/2}\cdot\left(3x^2+5+2^x\cdot\ln 2\right)$

⑥ $f(x) = x^4 - 3x\big|_{x=1} = 1-3 = -4$; $f'(x) = 4x^3 - 3\big|_{x=1} = 1 \Rightarrow$ tan. line: $y = -4 + 1(x-1)$

⑦ a.) $\lim\limits_{x\to 0} \underset{(0/0)}{\frac{1-\cos^2 x}{3x^2}} \overset{L.H.}{=} \lim\limits_{x\to 0} \frac{-2\cdot\cos x(-\sin x)}{6x} = \lim\limits_{x\to 0} \frac{\sin 2x}{6x} = \frac{2}{6}$

b.) $\lim\limits_{x\to\infty} \underset{(\infty/\infty)}{\frac{x^2-5x+12}{x-3x^2}} \overset{L.H.}{=} \lim\limits_{x\to\infty} \underset{(\infty/\infty)}{\frac{2x-5}{1-6x}} \overset{L.H.}{=} \lim\limits_{x\to\infty} \frac{2}{-6} = -\frac{1}{3}$

c.) $\lim\limits_{x\to 2} \underset{(0/0)}{\frac{x^2-4}{x^2-5x+6}} \overset{L.H.}{=} \lim\limits_{x\to 2} \frac{2x}{2x-5} = \frac{4}{-1} = -4$; d.) $\lim\limits_{x\to 0^+} \underset{(\infty/\infty)}{\frac{\ln x}{x^{-2}}} \overset{L.H.}{=} \lim\limits_{x\to 0^+} \frac{\frac{1}{x}}{-2x^{-3}} = \lim\limits_{x\to 0^+} \frac{x^{\cancel{3}\,2}}{-2\cancel{x}} = 0^-$

⑨ $f(x) = x^4 - 10x^3 + 36x^2 + 5$   $D_f = (-\infty;\infty)$

$f'(x) = 4x^3 - 30x^2 + 72x$

$f''(x) = 12x^2 - 60x + 72 = 12\cdot(x^2-5x+6) = 12(x-2)(x-3) = 0$ if $x=2$ or $x=3$

| $x$ | $(-\infty;2)$ | $2$ | $(2;3)$ | $3$ | $(3;\infty)$ |
|---|---|---|---|---|---|
| $f''(x)$ | $+$ | $0$ | $-$ | $0$ | $+$ |
| $f(x)$ | $\cup$ | $85$ infl | $\cap$ | $59$ infl | $\cup$ |

$f(2) = 16 - 80 + 144 + 5 = 85$

$f(3) = 81 - 270 + 243 + 5 = 59$

⑧ $f(x) = x^2 \cdot \ln x$ ; $D_f = (0;\infty)$

$f'(x) = 2x\cdot\ln x + \frac{1}{x}\cdot x^2 = 2x\cdot\ln x + x = x\cdot(2\ln x+1) = 0$

iff $2\ln x + 1 = 0$

$\ln x = -1/2 \Rightarrow x = e^{-1/2} = \frac{1}{\sqrt{e}}$

| $x$ | $(0; 1/\sqrt{e})$ | $1/\sqrt{e}$ | $(1/\sqrt{e};\infty)$ |
|---|---|---|---|
| $f'(x)$ | $-$ | $0$ | $+$ |
| $f(x)$ | $\searrow$ | $-1/2e$ min | $\nearrow$ |

⑩ $f(x) = x^2\cdot\ln x$ ; $D_f = (0;\infty)$

$\lim\limits_{x\to 0^+} \underset{(0\cdot\infty)}{x^2\cdot\ln x} = \lim\limits_{x\to 0^+} \underset{(\infty/\infty)}{\frac{\ln x}{x^{-2}}} = 0^-$ (by L'Hospital, see 7/d.)

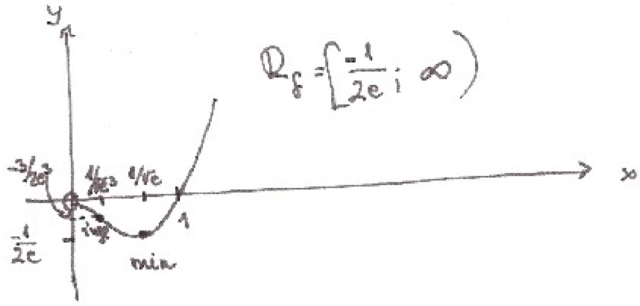
$\lim\limits_{x\to\infty} \underset{(\infty\cdot\infty)}{x^2\cdot\ln x} = \infty$

$f'(x) = x\cdot(2\ln x+1)$ ; monotonity, extrema see in Ex. 8)

$f''(x) = 2\ln x + 1 + x\cdot\frac{2}{x} = 2\ln x + 3 = 0 \Rightarrow \ln x = -3/2 \Rightarrow x = e^{-3/2} = \frac{1}{(\sqrt{e})^3}$

| $x$ | $(0; 1/\sqrt{e^3})$ | $1/(\sqrt{e})^3$ | $(1/\sqrt{e^3};\infty)$ |
|---|---|---|---|
| $f''(x)$ | $-$ | $0$ | $+$ |
| $f(x)$ | $\cap$ | $\frac{-3}{2e^3}$ infl | $\cup$ |

$f(1) = 0$




$R_f = \left[\frac{-1}{2e};\infty\right)$

⑪ a.) $\begin{vmatrix} 2 & x & 1 \\ 0 & 1 & 1 \\ 3 & 0 & 2 \end{vmatrix} = 2\cdot\underbrace{\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix}}_{2} - x\underbrace{\begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix}}_{-3} + 1\cdot\underbrace{\begin{vmatrix} 0 & 1 \\ 3 & 0 \end{vmatrix}}_{-3} = 4+3x-3 = 1+3x = 5$

$3x = 4$

$x = 4/3$

b.) $\underline{a} = 3\underline{i} + 2\underline{j} + \underline{k}$ $\quad |\underline{a}| = \sqrt{9+4+1} = \sqrt{14}$ ; $\underline{a}\cdot\underline{b} = 12-2+5 = 15$

$\underline{b} = 4\underline{i} - \underline{j} + 5\underline{k}$ $\quad |\underline{b}| = \sqrt{16+1+25} = \sqrt{42}$

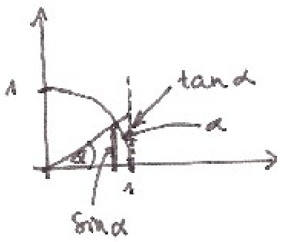
$\Downarrow$

$\cos\varphi = \frac{15}{\sqrt{14}\cdot\sqrt{42}} = \frac{15}{7\cdot\sqrt{12}}$

$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 2 & 1 \\ 4 & -1 & 5 \end{vmatrix} = \underline{i}\underbrace{\begin{vmatrix} 2 & 1 \\ -1 & 5 \end{vmatrix}}_{11} - \underline{j}\underbrace{\begin{vmatrix} 3 & 1 \\ 4 & 5 \end{vmatrix}}_{11} + \underline{k}\underbrace{\begin{vmatrix} 3 & 2 \\ 4 & -1 \end{vmatrix}}_{-11} = 11\underline{i} - 11\underline{j} - 11\underline{k}$

## Theoretical questions

1.)




$$\sin\alpha \le \alpha \le \tan\alpha = \frac{\sin\alpha}{\cos\alpha} \qquad / : \sin\alpha$$

$$\Downarrow$$

$$1 \le \frac{\alpha}{\sin\alpha} \le \frac{1}{\cos\alpha} \quad \text{and } \alpha \to 0^+$$

$$\frac{1}{\cos 0} = 1$$

$1 = 1$

$\frac{\alpha}{\sin\alpha} \to 1$ by the pinching theorem (policemen)

2.) $f(x)$ is differentiable at $x = x_0$ means, that $\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$ = exists, and it is finite.

Because the denominator, $h \to 0$, the existing finite limit of the fraction requires for the numerator: $f(x_0+h) - f(x_0) \to 0$

That means, $\lim_{h\to 0} f(x_0+h) = f(x_0)$, so $f$ is continuous at $x_0$.

3.)
$$[f(x) + g(x)]' = \lim_{h\to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} = \lim_{h\to 0}\Bigg[\underbrace{\frac{f(x+h)-f(x)}{h}}_{f'(x)} + \underbrace{\frac{g(x+h)-g(x)}{h}}_{g'(x)}\Bigg]$$

(because $f, g$ are differentiable)

4.)
$$[\tan(\tan^{-1}x)]' = \underbrace{\frac{1}{\cos^2(\tan^{-1}x)} \cdot [\tan^{-1}(x)]'}_{} = x' = \underbrace{1}_{}$$

So
$$(\tan^{-1}x)' = \cos^2(\tan^{-1}x) = \frac{\cos^2(\tan^{-1}x)}{\cos^2(\tan^{-1}x) + \sin^2(\tan^{-1}x)} = \frac{1}{1+\tan^2(\tan^{-1}x)} = \frac{1}{1+x^2}$$

5.)
$$\cosh^2 x - \sinh^2 x = \left[\frac{1}{2}(e^x + e^{-x})\right]^2 - \left[\frac{1}{2}(e^x - e^{-x})\right]^2 =$$

$$= \frac{1}{4}\left[\cancel{e^{2x}} + 2\underbrace{e^x e^{-x}}_{1} + \cancel{e^{-2x}} - (\cancel{e^{2x}} - 2\underbrace{e^x e^{-x}}_{1} + \cancel{e^{-2x}})\right] = \frac{1}{4}(2+2) = 1$$