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Graphs with no induced C_4 and $2K_2$

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Abstract.

We characterize the structure of graphs containing neither the 4-cycle nor its complement as an induced subgraph. This self-complementary class $\mathcal{G}$ of graphs includes split graphs, which are graphs whose vertex set is the union of a clique and an independent set. In the members of $\mathcal{G}$, the number of cliques (as well as the number of maximal independent sets) cannot exceed the number of vertices. Moreover, these graphs are almost extremal to the theorem of Nordhaus and Gaddum (1956).

1. Results.

We study undirected graphs without loops and multiple edges. A graph G is called *F-free* if no *induced subgraph* of G is isomorphic to F. In this note we give the structural characterization of the self-complementary class $\mathcal{G}$ of graphs which are C_4-free and $2K_2$-free, where C_4 denotes the cycle of four vertices and $2K_2$ is the matching of two edges, i.e., the complement of C_4.

Theorem 1.1. *A graph $G = (V, E)$ is C_4-free and $2K_2$-free if and only if there is a partition $V_1 \cup V_2 \cup V_3 = V$ with the following properties*

(i) V_1 is an independent set in G.

(ii) V_2 is the vertex set of a complete subgraph in G.

(iii) $V_3 = \emptyset$ or $|V_3| = 5$, and in the latter case V_3 induces a 5-cycle in G.

(iv) If $V_3 \neq \emptyset$, then for all $v_i \in V_i$, $i = 1, 2, 3$, $v_1v_3 \notin E$ and $v_2v_3 \in E$ hold.

A *split graph*, as introduced in [3, 5], is a graph satisfying properties (i) and (ii) of Theorem 1.1 with $V_1 \cup V_2 = V$. Hence, $\mathcal{G}$ is a natural extension of the thoroughly investigated class of split graphs. On the other hand, $\mathcal{G}$ is a subclass of pseudothreshold graphs characterized by Chvátal and Hammer in [1, Theorem 4(iii)] as the graphs $G = (V, E)$ admitting a vertex partition $V_1 \cup V_2 \cup V_3 = V$ that satisfies (i), (ii), (iv), and the further property that no three vertices in V_3 are pairwise non-adjacent.

Corollary 1.2. *If $G = (V, E)$ is a C_4-free and $2K_2$-free graph, then:*

(i) either G is a split graph, or there are exactly five distinct vertices $v_i \in V$, $i = 1, \ldots, 5$, such that each $G - v_i$ is a split graph,

(ii) G is a pseudothreshold graph.

Recently, Prömel and Steger [7] proved the following closely related asymptotic result: the ratio of the numbers of split graphs and C_4-free graphs on n vertices tends to 1 as n tends to infinity.

Split graphs are perfect (in fact, split graphs form the largest self-complementary class consisting of chordal graphs); however, not all graphs in $\mathcal{G}$ are perfect since C_5 is in $\mathcal{G}$. On the other hand, the following simple corollary may be of some interest because Berge's famous conjecture concerning perfect graphs has not yet been verified for C_4-free (or, equivalently, for $2K_2$-free) graphs.

Corollary 1.3 *The Strong Perfect Graph Conjecture is true for the class of C_4-free and $2K_2$-free graphs.*

The following important property of graphs $G \in \mathcal{G}$ shows some similarity to the class of chordal graphs.

Corollary 1.4. *If $G = (V, E)$ is a C_4-free and $2K_2$-free graph, then it contains at most $|V|$ maximal independent sets and at most $|V|$ cliques.*

The well-known theorem of Nordhaus and Gaddum [6] states that for any graph $G = (V, E)$, $\chi(G) + \chi(\bar{G}) \leq |V| + 1$ holds, where $\bar{G}$ and $\chi(G)$ denote the complement and the chromatic number of G, respectively. The following corollary to Theorem 1.1 shows that the members of $\mathcal{G}$ are almost extremal with respect to this theorem.

Corollary 1.5. *If a graph $G = (V, E)$ is C_4-free and $2K_2$-free, then $\chi(G) + \chi(\bar{G}) \geq |V|$.*

Wagon [9] proved that if $G = (V, E)$ is a $2K_2$-free graph with maximum clique size $\omega(G)$, then $\chi(G) \leq \binom{\omega(G)+1}{2}$.
For C_4-free and $2K_2$-free graphs the following stronger upper bound is valid. It was previously proved by Gyárfás in [4] with a decomposition result weaker than our Theorem 1.1.

Corollary 1.6. *If a graph $G = (V, E)$ is C_4-free and $2K_2$-free, then $\chi(G) \leq \omega(G) + 1$. Here equality holds if and only G is not a split graph.*

Finally, we give a result related to a general problem due to Erdős et al.[2].

Theorem 1.7. *Let $k = 2, 3$ or 4. If $G = (V, E)$ is a C_4-free and $2K_2$-free graph without isolated vertices such that every edge is contained in a complete subgraph on k vertices, then there is a set of at most $|V|/k$ vertices that meets all cliques of G, unless G is isomorphic to C_5 (and $k = 2$).*

An example given in [8] shows that Theorem 1.7 does not hold for $k = 5$, even if G is assumed to be C_5-free. On the other hand, for $k = 2$ and 3, the conclusion holds for chordal graphs as well; see [8].

2. Proofs

Proof of Theorem 1.1. One can check that all graphs having a vertex partition with properties (i)–(iv) are C_4-free and $2K_2$-free. To prove the converse statement, suppose that $G = (V, E)$ contains neither C_4 nor $2K_2$ as an induced subgraph. If G does *not* contain an induced cycle or its complement on more than three vertices, then G is a split graph (by the results of [3, 5], i.e., it satisfies the requirements with $V_3 = \emptyset$. Otherwise, since $\mathcal{G}$ is self-complementary, we can assume without loss of generality that C_k is an induced cycle on $k \geq 4$ vertices in G. Now $k = 4$ is impossible for G is C_4-free, and $k \geq 6$ is excluded since every cycle of length at least six contains $2K_2$ as an induced subgraph. Thus, $k = 5$, i.e., G has an induced 5-cycle.

Let $V_3 = \{v_1, \dots, v_5\}$ be the vertex set of this 5-cycle, where $v_i v_j \in E$ if and only if $|j - i| = 1$ or 4. Let x denote an arbitrary vertex in $V - V_3$. We claim that x is adjacent to at least one v_i if and only if x is adjacent to each v_i $(i = 1, \dots, 5)$.

Suppose to the contrary that, say, $xv_1 \in E$ and $xv_5 \notin$E. Since $\{x, v_1, v_3, v_4\}$ cannot induce $2K_2$, x must be adjacent to v_3 or v_4. Since $\{x, v_1, v_4, v_5\}$ cannot induce C_4, we obtain that $xv_4 \notin E$; thus $xv_3 \in E$. In this case, however, either $\{x, v_2, v_4, v_5\}$ induces $2K_2$ or $\{x, v_1, v_2, v_3\}$ induces C_4 (according as $xv_2 \in E$ or $xv_2 \notin E$), yielding a contradiction that proves our claim.

Define V_2 as the set of all vertices in $V \backslash V_3$ which are adjacent to at least one (or, equivalently, to each) $v_i \in V_3$, and let $V_1 = V \backslash (V_2 \cup V_3)$. For any $xy \in E$ induced by $V_1, \{x, y, v_1, v_2\}$ would induce $2K_2$, and for any $x, y \in V_2, x \neq y, xy \notin E$, the vertex set $\{x, y, v_1, v_3\}$ would induce C_4. Thus, V_2 induces a complete subgraph and V_1 is an independent set, completing the proof of Theorem 1.1. □

Proof of Corollary 1.3. If $G \in \mathcal{G}$ is not perfect, then G is not a split graph, and thus by Theorem 1.1 it contains an induced 5-cycle. □

Below we use the notation introduced in Theorem 1.1 and its proof.

Proof of Corollary 1.4. Since $\mathcal{G}$ is a self-complementary class, it suffices to prove that for any member $G = (V, E)$ of $\mathcal{G}$, G contains at most $|V|$ cliques. This is trivial if G is a split graph. Hence, we may assume that $|V_3| = 5$. Now observe that those cliques of G which meet V_3 are exactly those subgraphs of G which are induced by some set $V_2 \cup \{v_i, v_j\}$ where $|j - i| = 1$ or 4. There are five sets of this type, and the number of cliques of the split graph induced by $V_1 \cup V_2$ is at most $|V_1 \cup V_2|$. Therefore, the total number of cliques in G is at most $|V_1| + |V_2| + 5 = |V|$. □

Proof of Corollary 1.5. For any member $G = (V, E)$ of $\mathcal{G}$, it is obvious that $\chi(G) \geq |V_2|$ and $\chi(\bar{G}) \geq |V_1|$. Therefore we are home if $V_3 = \emptyset$. We may assume that $V_3 \neq \emptyset$. Let F and H denote the subgraphs of G induced by $V_2 \cup V_3$ and $V_1 \cup V_3$, respectively. Observe that $\chi(F) = |V_2| + 3$ and $\chi(\bar{H}) = |V_1| + 3$. Therefore,

$$\chi(G) + \chi(\bar{G}) \geq |V_2| + 3 + |V_1| + 3 = |V| + 1. \quad \square$$

Proof of Theorem 1.7. For split graphs (i.e., where $V_3 = \emptyset$) the result was proved in [8]. Hence, assume that $|V_3| = 5$. Note further that $V_2 = \emptyset$ implies $V_1 = \emptyset$ and $G = C_5$, so that $k = 2$ and three vertices can meet all edges of G.

Suppose that $V_2 \neq \emptyset$. Then the cliques meeting V_3 contain all vertices of V_2, therefore any $x \in V_2$ meets each of them. Moreover, the cliques disjoint from V_3 are cliques in the split graph induced by $V_1 \cup V_2$, too, and consequently, there is a set X of cardinality at most $(|V| - 5)/k$ that meets each of them. We conclude that the set $X \cup \{x\}$ meets all cliques of G, and

$$|X \cup \{x\}| \leq (|V| - 5)/k + 1 < |V|/k \text{ for } k \leq 4. \quad \square$$

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