Matematika III. – 2. gyak.

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1. Nevezetes függvények integráljai

1.1. Tétel

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$$

$$\int \frac{1}{x} \, dx = ln|x| + c$$

$$\int e^x \, dx = e^x + c$$

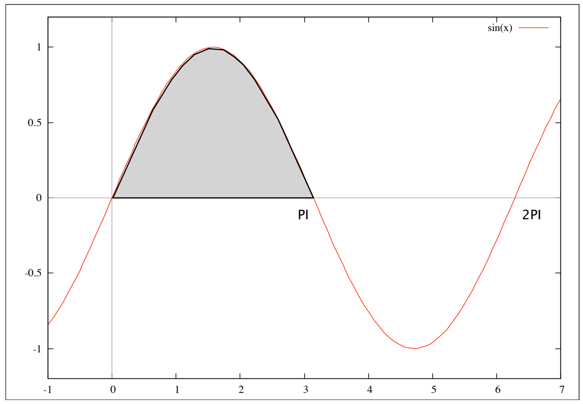
$$\int \sin(x) \, dx = -\cos(x) + c$$

$$\int \cos(x) \, dx = \sin(x) + c$$

$$\int \frac{1}{\cos^2 x} \, dx = \tan(x) + c$$

$$\int \frac{1}{\sin^2 x} \, dx = -\cot(x) + c$$

1.2. Példák



$$\begin{aligned}
\int_0^{\pi} \sin x\, dx &= [-\cos(x)]_0^{\pi} && (1)\\
&= (-cos\pi) - (-cos0) && (2)\\
&= (-(-1)) - (-1) && (3)\\
&= (1+1) = 2 && (4)
\end{aligned}$$

$$\begin{aligned}
\int_0^{2\pi} \sin x\, dx &= [-\cos(x)]_0^{2\pi} && (5)\\
&= [-\cos(2\pi)] - [-\cos(0)] && (6)\\
&= -1 - [-1] = 0 && (7)
\end{aligned}$$

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos x\, dx &= [\sin(x)]_{-\pi}^{\pi} && (8)\\
&= [\sin(\pi)] - [\sin(-\pi)] && (9)\\
&= 0 - 0 = 0 && (10)
\end{aligned}$$

1. $\int 2\sin x\, dx = 2 \cdot -\cos(x) + c$
2. $\int 3\cos x - \sin x\, dx = 3\int \cos x\, dx - \int \sin x\, dx = 3\sin x - (-\cos x) + c = 3\sin x + \cos x) + c$
3. $\int \frac{\cos x}{3}\, dx = \frac{1}{3}\sin x + c$

1.3. Tétel

$$\int f(x)\,dx = F(x) + c$$

$$\int f(ax+b)\,dx = \frac{F(ax+b)}{a} + c$$

4. $\int \sin(6x+4) = \frac{-\cos(6x+4)}{6} + c$

5. $\int \cos(-4-5x) = \frac{\sin(-4-5x)}{5} + c$

6. $\int \frac{1}{\sin^2(3x+2)} = -\frac{\cot(3x+2)}{3} + c$

7. $\int \frac{1}{\cos^2(-6x+4)} = \frac{\tan(-6x+4)}{-6} + c$

8. $\int e^{2x}\,dx = \frac{e^{2x}}{2} + c$

9. $\left(\frac{e^{2x}}{2}\right)' = \frac{1}{2} \cdot e^{2x} \cdot 2 = e^{2x}$

10. $\int_{-1}^{1} e^x\,dx = [e^x]_{-1}^{1} = e^1 - e^{-1} = e - \frac{1}{e} + c$

11. $\int_1^3 \frac{1}{x}\,dx = [\ln|x|]_1^3 = \ln 3 - \ln 1 = \ln(\frac{3}{1}) = \ln 3$
Átnézni: Logaritmus azonosságait!!

12. $\int_1^3 (3x + \frac{1}{x^2})\,dx = ?$

$$\begin{aligned}
\int_1^3 (3x + \frac{1}{x^2})\,dx &= [\frac{3x^2}{2} + \frac{x^{-1}}{-1}]_1^3 \\
&= [\frac{3x^2}{2} - x^{-1}]_1^3 \\
&= \frac{3 \cdot 3^2}{2} - 3^{-1} - (\frac{3 \cdot 1^2}{2} - 1^{-1}) \\
&= \frac{27}{2} - \frac{1}{3} - \frac{3}{2} + 1 \\
&= \frac{81}{6} - \frac{2}{6} - \frac{9}{6} + \frac{6}{6} \\
&= \frac{76}{6} = \frac{38}{3}
\end{aligned}$$

13. $\int_0^{\frac{\pi}{2}} (\sin x + \cos x)\, dx = ?$

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} (\sin x + \cos x)\, dx &= \big[-\cos x + \sin x\big]_0^{\frac{\pi}{2}} \\
&= \big[-\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\big] - \big[-\cos 0 + \sin 0\big] \\
&= \big[-0 + 1\big] - \big[-1 + 0\big] \\
&= 1 + 1 = 2
\end{aligned}$$

1.4. Emlékeztető: összetett függvények deriválása

$$\Big(f\big(g(x)\big)\Big)' = f'\big(g(x)\big) \cdot g'(x)$$

Példa:

$$f\big(g(x)\big) = (x+2)^2$$

Ebben az esetben:

$$f(x) = X^2 \quad g(x) = x + 2$$

Tehát

$$\Big(f\big(g(x)\big)\Big)' = ((x+2)^2)' = 2(x+2) \cdot 1 = 2x + 4$$

1.5. Tétel

$$\int f^n \cdot f'\, dx = \frac{f^{n+1}}{n+1} + c$$

Ennek a speciális esete, amikor $n = 1$:

$$\int f \cdot f'\, dx = \frac{f^2}{2} + c$$

Az összefüggést így láthatjuk be:

$$(\frac{f^{n+1}}{n+1})' = \frac{(n+1)f^n \cdot f'}{n+1} = f^n \cdot f'$$

14. $\int \sin x \cdot \cos x\, dx = \frac{\sin^2 x}{2} + c$

$$f(x) = \sin x \quad f'(x) = \cos x$$

15. $\int \frac{\ln x}{x}\,dx = \frac{\ln^2 x}{2} + c$

$$f(x) = \ln x \quad f'(x) = \frac{1}{x}$$

16. $\int x^2(2x^3+4)\,dx = ?$

$$\begin{aligned}\int x^2(2x^3+4)\,dx &= \frac{1}{6}\int 6x^2(2x^3+4)\,dx \\ &= \frac{1}{6}(2x^3+4)^2\end{aligned}$$

$$f(x) = 2x^3+4 \quad f'(x) = 6x^2$$

17. $\int (2x^3+4)^5 \cdot x^2\,dx = ?$

$$\begin{aligned}\int (2x^3+4)^5\,dx &= \frac{1}{6}\int 6x^2(2x^3+4)^5\,dx \\ &= \frac{1}{6}\frac{(2x^3+4)^6}{6} + c \\ &= \frac{1}{36}(2x^3+4)^6 + c\end{aligned}$$

$$f(x) = 2x^3+4 \quad f'(x) = 6x^2$$

18. $\int x^2 \cdot \sqrt{6x^3+4}\,dx = ?$

$$\begin{aligned}\int x^2 \cdot \sqrt{6x^3+4}\,dx &= \frac{1}{18}\int (6x^3+4)^{\frac{1}{2}} \cdot 18x^2\,dx \\ &= \frac{1}{18}\frac{(6x^3+4)^{\frac{3}{2}}}{\frac{3}{2}} \\ &= \frac{2}{54} \cdot \sqrt{(6x^3+4)^3} + c\end{aligned}$$

$$f(x) = 6x^3+4 \quad f'(x) = 18x^2$$

19. $\int e^x \cdot (1-e^x)\,dx = ?$

$$\begin{aligned}\int e^x \cdot (1-e^x)\,dx &= -\int -e^x \cdot (1-e^x)\,dx \\ &= -\frac{(1-e^x)^2}{2} \\ &= -\frac{1}{2}(1-2e^x+e^{2x}) + c\end{aligned}$$

$$f(x) = 1-e^x \quad f'(x) = -e^x$$

20. $\int \sin^4 x \cdot \sin 2x\,dx = ?$

$$\sin 2x = 2\sin x \cos x$$

$$\begin{aligned}\int \sin^4 x \cdot \sin 2x \, dx &= \int \sin^4 x \cdot 2 \sin x \cos x \, dx \\ &= 2 \int \sin^5 x \cdot \cos x \, dx \\ &= 2 \cdot \frac{\sin^6 x}{6} + c \\ &= \frac{\sin^6 x}{3} + c\end{aligned}$$

$$f(x) = \sin x \quad f'(x) = \cos x$$

21. $\int \frac{x}{\sqrt{x^2+6}} \, dx = ?$

$$\begin{aligned}\int \frac{x}{\sqrt{x^2+6}} \, dx &= \int x \cdot (x^2+6)^{-\frac{1}{2}} \, dx \\ &= \frac{1}{2} \int 2x \cdot (x^2+6)^{-\frac{1}{2}} \, dx \\ &= \frac{1}{2} \frac{(x^2+6)^{\frac{1}{2}}}{\frac{1}{2}} \\ &= \frac{1}{2} \cdot 2 \cdot (x^2+6)^{\frac{1}{2}} \\ &= (x^2+6)^{\frac{1}{2}} = \sqrt{x^2+6} + c\end{aligned}$$

$$f(x) = x^2 + 6 \quad f'(x) = 2x$$

22. $\int \frac{\sin x}{\sqrt[3]{\cos^2 x}} \, dx = ?$

$$\begin{aligned}\int \frac{\sin x}{\sqrt[3]{\cos^2 x}} \, dx &= \int \sin x \cos^{-\frac{2}{3}} x \, dx \\ &= -\int -\sin x \cos^{-\frac{2}{3}} x \, dx \\ &= -\frac{\cos^{\frac{1}{3}} x}{\frac{1}{3}} \\ &= -3 \cos^{\frac{1}{3}} x + c = -3 \sqrt[3]{\cos x} + c\end{aligned}$$

$$f(x) = \cos x \quad f'(x) = -\sin x$$

23. $\int \frac{\sin^5 x}{\cos^7 x} \, dx = ?$

$$\begin{aligned}\int \frac{\sin^5 x}{\cos^7 x} \, dx &= \int \frac{\sin^5 x}{\cos^5 x \cdot \cos^2 x} \, dx \\ &= \int \operatorname{tg}^5 x \frac{1}{\cos^2 x} \, dx \\ &= \frac{\operatorname{tg}^6 x}{6} + c\end{aligned}$$

$$f(x) = \mathrm{tg}^5\, x \quad f'(x) = \frac{1}{\cos^2 x}$$

1.6. Tétel

$$\int \frac{f'(x)}{f(x)}\, dx = \ln |f(x)| + c$$

24. $\int \frac{2x}{x^2+7}\, dx =?$

$$\int \frac{2x}{x^2+7}\, dx \quad = \quad \ln |x^2+7| + c = \ln(x^2+7) + c$$

$$f(x) = x^2+7 \quad f'(x) = 2x$$

25. $\int \frac{5x^2}{x^3+4}\, dx =?$

$$\begin{aligned} \int \frac{5x^2}{x^3+4}\, dx &= \frac{5}{3} \int \frac{3x^2}{x^3+4}\, dx \\ &= \frac{5}{3} \ln |x^3+4| + c \\ &= \ln(|x^3+4|)^{\frac{5}{3}} + c \end{aligned}$$

$$f(x) = x^3+4 \quad f'(x) = 3x^2$$

Logaritmus azonosságait átnézni!!

26. $\int \frac{4 \sin x}{5 \cos x + 4}\, dx =?$

$$\begin{aligned} \int \frac{4 \sin x}{5 \cos x + 4}\, dx &= -4 \int \frac{-\sin x}{5 \cos x + 4}\, dx \\ &= -\frac{4}{5} \int \frac{-5 \sin x}{5 \cos x + 4}\, dx \\ &= -\frac{4}{5} \ln |5 \cos x + 4| + c \end{aligned}$$

$$f(x) = 5 \cos x + 4 \quad f'(x) = -5 \sin x$$

27. $\int \mathrm{tg}\, x\, dx =?$

$$\begin{aligned} \int \mathrm{tg}\, x\, dx &= \int \frac{\sin x}{\cos x}\, dx \\ &= -\int \frac{-\sin x}{\cos x}\, dx \\ &= -\ln |\cos x| + c \end{aligned}$$

$$f(x) = \cos x \quad f'(x) = -\sin x$$

28. $\int \frac{-\sin 2x}{5+\cos^2 x}\,dx = ?$

$$\begin{aligned} \int \frac{-\sin 2x}{5+\cos^2 x}\,dx &= \frac{1}{2}\int \frac{-2\sin 2x}{5+\cos^2 x}\,dx \\ &= \frac{1}{2}\ln|5+\cos^2 x| + c \end{aligned}$$

A megoldáshoz ki kell számítani f(x) deriváltját (összetett függvény)

$$f(x) = 5+\cos^2 x \quad f'(x) = ?$$

$$\begin{aligned} f'(x) &= (5+\cos^2 x)' \\ &= 5' + (\cos^2 x)' \\ &= 0 + 2\cos x \cdot (-\sin x) = -2\cos x \sin x = -2\sin 2x \end{aligned}$$

29. $\int \frac{1}{\cos^2 x \operatorname{tg} x}\,dx = ?$

$$\begin{aligned} \int \frac{1}{\cos^2 x \operatorname{tg} x}\,dx &= \int \frac{1}{\cos^2 x} \cdot \frac{1}{\operatorname{tg} x}\,dx \\ &= \int \frac{\frac{1}{\cos^2 x}}{\operatorname{tg} x}\,dx \\ &= \ln|\operatorname{tg} x| + c \end{aligned}$$

$$f(x) = \operatorname{tg} x \quad f'(x) = \frac{1}{\cos^2 x}$$

30. $\int \frac{1}{x \ln x}\,dx = ?$

$$\begin{aligned} \int \frac{1}{x\ln x}\,dx &= \int \frac{\frac{1}{x}}{\ln x}\,dx \\ &= ln|\ln x| + c \end{aligned}$$

$$f(x) = \ln x \quad f'(x) = \frac{1}{x}$$